



11.2 The adjoint of a real finite-dimensional linear map

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Introduction

- In this topic, we will
 - Find the adjoint for real matrices
 - It will be given the name of “transpose”
 - Consider the relationship between the rank of a matrix, and the rank of its transpose





Matrix-vector multiplication

- Given the linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$, then it is easiest to think of

$$A\mathbf{u} = u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n$$

- Thus, we may deduce the i^{th} entry of $A\mathbf{u}$ is given by

$$\begin{aligned}(A\mathbf{u})_i &= u_1(\mathbf{a}_1)_i + \cdots + u_n(\mathbf{a}_n)_i \\ &= u_1a_{i,1} + \cdots + u_na_{i,n} \\ &= \sum_{j=1}^n u_j a_{i,j}\end{aligned}$$





The inner product of $A\mathbf{u}$ and $\mathbf{v}$

- Thus, if $(A\mathbf{u})_i = \sum_{j=1}^n u_j a_{i,j}$,

$$\begin{aligned}\text{then } \langle A\mathbf{u}, \mathbf{v} \rangle &= \sum_{i=1}^m \left(\sum_{j=1}^n (u_j a_{i,j}) \right) v_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n (u_j a_{i,j}) v_i \right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m (u_j a_{i,j}) v_i \right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m (v_i a_{i,j}) u_j \right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m (v_i a_{i,j}) \right) u_j = \langle \mathbf{u}, A^* \mathbf{v} \rangle\end{aligned}$$





Interpreting the result

- Question: What is $\left(A^* \mathbf{v}\right)_j = \sum_{i=1}^m \left(v_i a_{i,j}\right)$ where j goes from 1 to n ?

- Rewriting it, we have

$$\left(A^* \mathbf{v}\right)_j = \sum_{i=1}^m v_i a_{i,j} = v_1 a_{1,j} + \cdots + v_m a_{m,j}$$

$$\left(A \mathbf{u}\right)_i = \sum_{j=1}^n u_j a_{i,j} = u_1 a_{i,1} + \cdots + u_n a_{i,n}$$

- This is a linear combination of the rows of A





The adjoint matrix

- Thus, it seems that
 - The dimensions of the adjoint must be $n \times m$
 - The $(i, j)^{\text{th}}$ entry of A^* is the $(j, i)^{\text{th}}$ entry of A
 - That is:

$$(A^*)_{i,j} = a_{j,i}$$

- For example,

$$A = \begin{pmatrix} 3.2 & 7.9 & 0.8 & -0.3 \\ -5.6 & 4.9 & -8.2 & 0.5 \\ 0.1 & -4.7 & 1.2 & -5.4 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 3.2 & -5.6 & 0.1 \\ 7.9 & 4.9 & -8.2 \\ 0.8 & -8.2 & 1.2 \\ -0.3 & 0.5 & -5.4 \end{pmatrix}$$





The transpose

- The word *transpose* can mean for the position of two objects to be switched
 - For finite-dimensional real matrices, the adjoint is that matrix that has each entry off the diagonal switched with its mirror image across the diagonal
 - Hence, for finite-dimensional real matrices, the adjoint is called the *transpose* and denoted A^T

$$A = \begin{pmatrix} 3.2 & 7.9 & 0.8 & -0.3 \\ -5.6 & 4.9 & -8.2 & 0.5 \\ 0.1 & -4.7 & 1.2 & -5.4 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 3.2 & -5.6 & 0.1 \\ 7.9 & 4.9 & -8.2 \\ 0.8 & -8.2 & 1.2 \\ -0.3 & 0.5 & -5.4 \end{pmatrix}$$





The transpose

- If a theorem refers to the adjoint of a linear map, if you are applying that to linear maps on finite dimensional real vector spaces, you may substitute the phrase “the transpose”
 - For example,
 - The transpose of the transpose of a matrix is the matrix itself
- It is important to remember that the defining feature of the transpose is how it interacts with the inner product:

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^T \mathbf{v} \rangle$$





Example

- Let's check if that's actually true:

$$A = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 2 & 4 & 2 & 1 \\ -2 & 1 & 5 & -3 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 4 & 1 \\ -1 & 2 & 5 \\ 2 & 1 & -3 \end{pmatrix} \quad \langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} \right\rangle = -6$$

$$\langle \mathbf{u}, A^T \mathbf{v} \rangle = \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -15 \\ -15 \\ 3 \end{pmatrix} \right\rangle = -6$$





Goal

- Our next goal will be to show that the $\text{rank}(A) = \text{rank}(A^T)$
 - To show this, we will start with one theorem, and then give a sketch of the proof





Goal

Theorem

Given a collection of vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbf{R}^n$ and a linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$,
if $A\mathbf{u}_1, \dots, A\mathbf{u}_k$ are linearly independent, then so are $\mathbf{u}_1, \dots, \mathbf{u}_k$

Proof:

Assume that $\mathbf{u}_1, \dots, \mathbf{u}_k$ are not linearly independent.
Then, there exists a non-zero linear combination of these vectors that equals the zero vector:

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0},$$

Thus, $A(\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k) = \alpha_1 A\mathbf{u}_1 + \dots + \alpha_k A\mathbf{u}_k = \mathbf{0}$,
thus, $A\mathbf{u}_1, \dots, A\mathbf{u}_k$ are not linearly independent. ■





Goal

Theorem

If $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$, then $\text{rank}(A) = \text{rank}(A^T)$.

Sketch of the proof:

First, $\langle A^T A \mathbf{u}, \mathbf{u} \rangle = \langle A \mathbf{u}, (A^T)^T \mathbf{u} \rangle = \langle A \mathbf{u}, A \mathbf{u} \rangle = \|A \mathbf{u}\|_2^2$,
so $\mathbf{u}$ is in the null space of A if and only if $\mathbf{u}$ is in the null space of $A^T A$.

Thus, $\text{rank}(A^T A) = \text{rank}(A)$.

Find a basis for the range of $A^T A$ which is a linearly independent subset of $\mathbf{R}^n$: $A^T A \mathbf{u}_1, \dots, A^T A \mathbf{u}_r$.

Thus, $A \mathbf{u}_1, \dots, A \mathbf{u}_r$ must be linearly independent in $\mathbf{R}^m$.

Thus, $\text{rank}(A) \geq \text{rank}(A^T)$.

Now go the other way to show $\text{rank}(A^T) \geq \text{rank}(A)$. ■





Adjoint versus transpose

- Remember, a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ is one that maps one vector space onto another

- The *adjoint* of a linear map is that linear map that satisfies

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^* \mathbf{v} \rangle$$

- For a linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$,

then A can be represented as an $m \times n$ matrix

- The *adjoint* of a matrix is the $n \times m$ matrix that is the matrix A *reflected* through the diagonal, and we call it *transpose*





Adjoint versus transpose

- The transpose is the action of reflecting a matrix through the diagonal
 - It happens that the transpose is the adjoint if the matrix is real
 - If the matrix has complex entries (that is, $A: \mathbb{C}^n \rightarrow \mathbb{C}^m$), we can still calculate the transpose, but the transpose is not the adjoint of that complex matrix





Summary

- Following this topic, you now
 - Understand that the adjoint of a real matrix is the transpose
 - The rank of a matrix equals the rank of its transpose





References

- [1] https://en.wikipedia.org/wiki/Hermitian_adjoint





Acknowledgments

None so far.





Colophon

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